Matrix Analysis of Three-Dimensional Elastic Media Small and Large Displacements

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The paper describes a broad generalization of the matrix displacement technique for analyzing three-dimensional elastic media, both in the small and large displacement ranges. The body is approximated by an assembly of tetrahedrons, for which a simplified kinematic pattern is prescribed. Using the novel device of a natural definition for component stresses, nodal loads, and the elemental stiffness, a concise expression is established for the Cartesian stiffness of an arbitrary tetrahedron, from which the stiffness of the complete system is obtained. The solution of the small displacement problem is then straightforward. To extend the theory to large displacements, the concept of an additional or geometrical stiffness is introduced which represents the effects of change of geometry on the equilibrium conditions. The specification of natural nodal loads is thereby most helpful, yielding the surprisingly simple result that the geometrical stiffness of a tetrahedron is identical to that of a six-bar pin-jointed framework, whose members form a geometrically equivalent tetrahedron. The large displacement problem may now be solved by a straightforward procedure based on a step by step linearization. The theory covers the joint presence of external loads and thermal strains.

Nomenclature

l_{ij}	= length of edge (i, j) of tetrahedron
$\mathbf{l}$	= $\{l_{ij}\}$ = (6×1) matrix of lengths l_{ij}
V	= volume of tetrahedron
ρ_{ij}	= elongation of edge (i, j)
$\boldsymbol{\rho}_N$	= $\{\rho_{ij}\}$ = (6×1) natural displacement matrix
P_{ij}	= nodal pair of forces along edge (i, j)
$\mathbf{P}_N$	= $\{P_{ij}\}$ = (6×1) natural force matrix
$\mathbf{k}_N$	= (6×6) natural stiffness of tetrahedron
$Oxyz$	= Cartesian coordinate system
$\boldsymbol{\psi}_m$	= $\{x\, y\, z\}_m$ = column matrix of coordinates of point m
$\boldsymbol{\rho}_i$	= $\{u\, v\, w\}_i$ = column matrix of Cartesian nodal displacements at i
$\mathbf{P}_i$	= $\{U\, V\, W\}_i$ = column matrix of Cartesian nodal forces at i
$\mathbf{c}_{ij}$	= (1×3) row matrix of direction cosines for edge (i, j)
$\mathbf{C}_N$	= $\lceil \mathbf{c}_{ij} \rfloor$ = diagonal supermatrix of $\mathbf{c}_{ij}$
$\mathbf{k}$	= (12×12) Cartesian stiffness of tetrahedron
$\mathbf{R}$	= $\{R\}$ = external nodal loads
$\mathbf{J}$	= $\{J\}$ = nodal loads arising from blocked thermal expansion
$\mathbf{K}$	= stiffness of assembled system
α; Θ	= coefficient of linear thermal expansion; temperature
$\mathcal{A}$	= transformation matrix
$\mathbf{D}$	= difference matrix
$\mathbf{a}$, $\mathbf{T}$, $\mathbf{M}$	= identification or location matrices
$\mathbf{I}$; $\mathbf{e}$ = $\{1\}$	= unit matrix; unit column matrix
E; ν	= Young's modulus; Poisson's ratio
σ; ϵ	= stress; strain
Δ	= increment
t	= transposed matrix
$\{\ldots\}$, $\lceil \ldots \rfloor$	= column matrix; diagonal matrix

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Introduction

THE considerable success achieved in the computer orientated analysis of arbitrary membrane structures by the matrix displacement method, in conjunction with a kinematic idealization of the system based on triangular elements, naturally leads us to suppose that the investigation of three-dimensional deformable bodies may prove equally fruitful by the adoption of tetrahedra as elements. Again, as in the two-dimensional case, we may use any suitable net of nodal points or vertices, the closeness of which may be adjusted to the expected stress gradients and the geometry of the boundary. We also observe that the postulation of linear variation of the displacements within each element uniquely determines the $3 \times 3 = 9$ open constants in terms of the 9 displacement components at the corresponding nodal points. This yields, as for the triangular element, a constant strain and stress field within each tetrahedron. It follows that extreme (e.g., needlelike) configurations should, in general, be avoided. The stiffness matrix of a tetrahedron in terms of the $4 \times 3 = 12$ Cartesian components of the nodal displacements is clearly of order 12×12.

Our foregoing comments may not be particularly original in the field of linear elasticity. (For example, Gallagher et al.[8] derived rather complex expressions for the full 12×12 stiffness matrix of an element, using the more traditional stiffness-displacement approach. This paper is valuable for some interesting two-dimensional applications exploring the applicability of the technique in the presence of inelastic deformations.) When we attempt, however, to extend the theory to more general problems of large displacements in the elastic and nonelastic regimes, we quickly realize that the standard concept of elemental stiffness, as used in the past, is not really appropriate and leads to elaborate and unsymmetric algebraic forms. A better and more concise approach has been initiated in Refs. 1 and 2 and developed there in some detail for flanges, beams, and triangular elements. A characteristic feature of this technique is the specification of an invariant or natural stiffness, which excludes the rigid body motions. Since the latter do not cause any straining of the element, they are, in principle, superfluous in setting up the stiffness matrix. Thus, for a triangle, this stiffness is seen to be merely of order 6 − 3

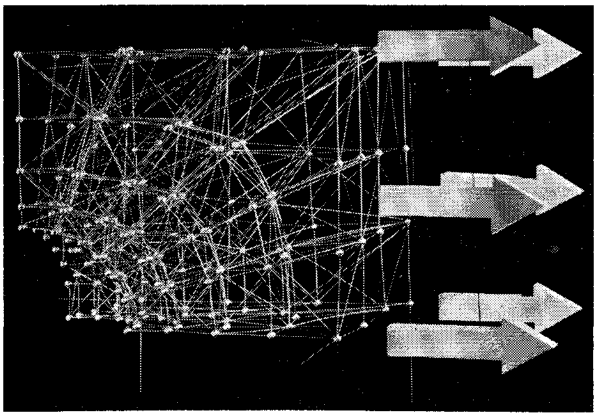

Fig. 1 Model of three-dimensional continuum with hole; one-eighth of space shown under uni-axial stress.

= 3. The derivation of the relevant formulas for a triangular element is greatly simplified by a novel definition of the stress and strain vectors in terms of components taken parallel to the three sides. Accordingly, the natural stiffness then relates the elongations and nodal forces along the sides. The simplicity of the method is clearly due to the adoption of force and displacement conventions that are in harmony with the given geometry.

However, the striking advantage of the invariant approach becomes only evident when formulating the large displacement analysis. Since the general philosophy of the method is developed in Refs. 1 and 2,† it suffices if we reproduce here merely the essence of the argument. Note, moreover, that the method bears certain affinities to that proposed in Refs. 3 and 4. In general, the technique involves a step by step process of sequential linearization corresponding to an incremental representation of the load and/or thermal history. For each step we obtain the accretion to the displacement vector from a linear matrix equation, which is formally identical to the small displacement equation, the only difference arising in the specification of the stiffness matrix, which must be considered as the sum of two component matrices. The first matrix is, in fact, the standard elastic stiffness. More interesting is the second one, which represents the effects of change of geometry on the equilibrium conditions and is denoted by us as the geometrical stiffness. It is found to be linearly dependent on the instantaneous natural nodal loads whose intensity derives from a summation process over the preceding "loading" steps. Thus, the new stiffness is seen to be akin to the restoring actions in a displaced string under tension. The formulation of the geometrical stiffness is greatly simplified, both from the conceptual and computational points of view, by the aforementioned natural nodal loads and displacements. In particular, for a tetrahedron element, the geometrical stiffness is obtained most simply from an equivalent model consisting of six bars, running along the edges of the tetrahedron and subject to constant loads.

The linear small displacement analysis is presented below under Sec. 1. Its generalization for large displacements is given in Sec. 2. Some interesting applications and a further generalization to cover nonelastic effects are to be discussed in a separate publication. The application of the theory on the digital computer is extremely simple, thanks to a sophisticated matrix scheme developed by the author's teams for the Remington Rand UNIVAC 1107 and the Ferranti (now I.C.T.) machines.

1. Small Deflexion Theory

1.1 The Displacement and Force Vectors for a Single Tetrahedron

It is evident that any three-dimensional continuum may be approximated or idealized by an assembly of, say s, tetrahedra (Fig. 1). Consider now the arbitrary tetrahedron element shown in Figs. 2a and 2b, where some of the relevant geometrical data are specified. The position of the tetrahedron is assumed to be fixed by the Cartesian coordinates (x_i, y_i, z_i) of the four vertices $i = 1, 2, 3, 4$. This information may be extracted from a $3n \times 1$ column matrix

$$\mathbf{\Psi} = \{\boldsymbol{\psi}_1\ \boldsymbol{\psi}_2 \ldots \boldsymbol{\psi}_m \ldots \boldsymbol{\psi}_n\} \tag{1}$$

where

$$\boldsymbol{\psi}_m = \boldsymbol{\phi}_m{}^t = \{x_m\ y_m\ z_m\} \tag{2}$$

The matrix $\mathbf{\Psi}$ may be stored in an appropriate part of the computer store. It should be noted that each nodal point is enumerated both in the unique sequence 1 to n, chosen to count the nodal points in the idealized structure, and in the specification 1 to 4, selected to define the vertices in the tetrahedron(a) to which the nodal point may appertain. To call upon the coordinates in groups corresponding to each of the s tetrahedrons we use a device described further below.

Following on our comments in the Introduction and the notation of Fig. 3, we introduce the two (6×1) vectors

$$\boldsymbol{\rho}_N = \{\rho_{12}\ \rho_{23}\ \rho_{34}\ \rho_{41}\ \rho_{13}\ \rho_{24}\} \tag{3}$$

$$\mathbf{P}_N = \{P_{12}\ P_{23}\ P_{34}\ P_{41}\ P_{13}\ P_{24}\} \tag{4}$$

which are sufficient to define completely the state of deformation and the nodal loads of each tetrahedron. Thus, ρ_{31} is the elongation of the edge (3, 1), while P_{23} is the self-equilibrating pair of forces in the direction of (2, 3) acting at the nodal points 2 and 3 (Figs. 3a and 3b). For the analysis of the as-

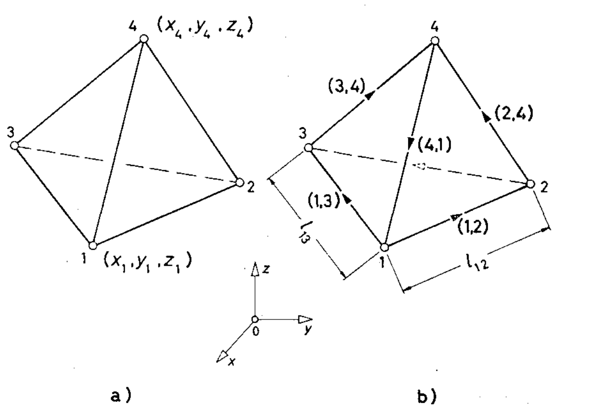

Fig. 2 Arbitrary tetrahedron element; notation of vertices, edges, and their assigned positive directions.

† See also Refs. 6 and 7.

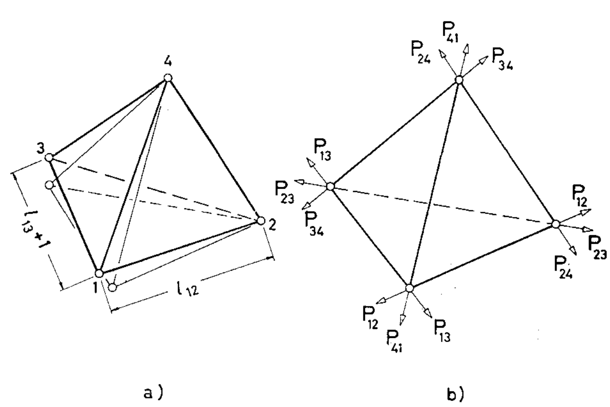

Fig. 3 Natural kinematic mode $\rho_{31} = 1$; natural nodal loads.

sembled idealized system, we also require the more conventional (12 × 1) Cartesian representation of the nodal displacement and force vectors of an element (Figs. 4a and 4b). Two different sequential orders may thereby be of importance and are quoted in turn below. The first pair of vectors is

$$\bar{\boldsymbol{\rho}} = \{\boldsymbol{\rho}_1\ \boldsymbol{\rho}_2\ \boldsymbol{\rho}_3\ \boldsymbol{\rho}_4\} \tag{5}$$

$$\bar{\mathbf{P}} = \{\mathbf{P}_1\ \mathbf{P}_2\ \mathbf{P}_3\ \mathbf{P}_4\} \tag{6}$$

where $\boldsymbol{\rho}_i$, $\mathbf{P}_i$ are the (3 × 1) vectors

$$\boldsymbol{\rho}_i = \{u_i\ v_i\ w_i\} \tag{7}$$

$$\mathbf{P}_i = \{U_i\ V_i\ W_i\} \tag{8}$$

of the Cartesian components of the nodal displacements or forces. The second set of (12 × 1) vectors is written as

$$\boldsymbol{\rho} = \{\mathbf{u}\ \mathbf{v}\ \mathbf{w}\} \tag{9}$$

$$\mathbf{P} = \{\mathbf{U}\ \mathbf{V}\ \mathbf{W}\} \tag{10}$$

Here each of the submatrices is a (4 × 1) vector incorporating the associated component scalars at the four vertices. For example,

$$\mathbf{v} = \{v_1\ v_2\ v_3\ v_4\} \tag{11}$$

$$\mathbf{W} = \{W_1\ W_2\ W_3\ W_4\} \tag{12}$$

It should be noted that the displacement vector of Eq. (5) or Eq. (9) includes the rigid body motions, which are excluded in the natural vector of Eq. (3). Correspondingly, the force vector of Eq. (6) or Eq. (10) ignores the equilibrium conditions, which are allowed for in Eq. (4).

In accordance with the theory of Ref. 5, alternative presentations of associated kinematic and force vectors imposed on a body are always related by dual matrix transformations. Thus, we have

$$\bar{\boldsymbol{\rho}} = \mathbf{T}\boldsymbol{\rho} \qquad \mathbf{P} = \mathbf{T}^t\bar{\mathbf{P}} \tag{13}$$

where, in the present case,

$$\mathbf{T} = \begin{bmatrix} \mathbf{E}_{11} & \mathbf{E}_{21} & \mathbf{E}_{31} \\ \mathbf{E}_{12} & \mathbf{E}_{22} & \mathbf{E}_{32} \\ \mathbf{E}_{13} & \mathbf{E}_{23} & \mathbf{E}_{33} \\ \mathbf{E}_{14} & \mathbf{E}_{24} & \mathbf{E}_{34} \end{bmatrix} \tag{14}$$

In Eq. (14) the $\mathbf{E}_{ij}$ are (3 × 4) matrices in which the only nonzero element is a unit at the position (i, j). Note that $\mathbf{T}$ is orthonormal, i.e.,

$$\mathbf{T}^t\mathbf{T} = \mathbf{I}_{12} \qquad \mathbf{T}^t = \mathbf{T}^{-1} \tag{15}$$

In order to construct the transformation matrix connecting the sets $\boldsymbol{\rho}_N$, $\mathbf{P}_N$ and $\bar{\boldsymbol{\rho}}$, $\bar{\mathbf{P}}$, it is first necessary to establish the direction cosines of the six edges of the tetrahedron. The selected positive directions of the vectors are indicated in Fig. 2b. We now write the direction cosines of each edge (i, j) as the (1 × 3) row matrix

$$\mathbf{c}_{ij} = [\cos(ij, x)\quad \cos(ij, y)\quad \cos(ij, z)] \tag{16}$$

Using Eq. (2) and the notation

$$\boldsymbol{\phi}_{ij} = \boldsymbol{\phi}_j - \boldsymbol{\phi}_i \tag{2a}$$

we have

$$\mathbf{c}_{ij} = (1/l_{ij})\boldsymbol{\phi}_{ij} \tag{17}$$

where l_{ij} is the length of the edge (i, j).

It is convenient to assemble the six matrices corresponding to Eq. (17) as the 6 × 6 diagonal super-matrix, each element of which is a (1 × 3) row matrix. Thus,

$$\mathbf{C}_N = \lceil \mathbf{c}_{12}\ \mathbf{c}_{23}\ \mathbf{c}_{34}\ \mathbf{c}_{41}\ \mathbf{c}_{13}\ \mathbf{c}_{24} \rfloor \tag{18}$$

We also define a so-called difference matrix

$$\mathbf{D} = \begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{O}_3 & \mathbf{O}_3 \\ \mathbf{O}_3 & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{O}_3 \\ \mathbf{O}_3 & \mathbf{O}_3 & -\mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{I}_3 & \mathbf{O}_3 & \mathbf{O}_3 & -\mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{O}_3 & \mathbf{I}_3 & \mathbf{O}_3 \\ \mathbf{O}_3 & -\mathbf{I}_3 & \mathbf{O}_3 & \mathbf{I}_3 \end{bmatrix} \tag{19}$$

of dimensions 18 × 12.

It is now easy to confirm the relation

$$\boldsymbol{\rho}_N = \mathbf{C}_N\mathbf{D}\bar{\boldsymbol{\rho}} = \mathbf{C}_N\mathbf{D}\mathbf{T}\boldsymbol{\rho} \tag{20}$$

where the second formula follows from the first of Eqs. (13). The corresponding expressions for the force vectors are clearly

$$\mathbf{P} = \mathbf{T}^t\bar{\mathbf{P}} = \mathbf{T}^t\mathbf{D}^t\mathbf{C}_N{}^t\mathbf{P}_N \tag{21}$$

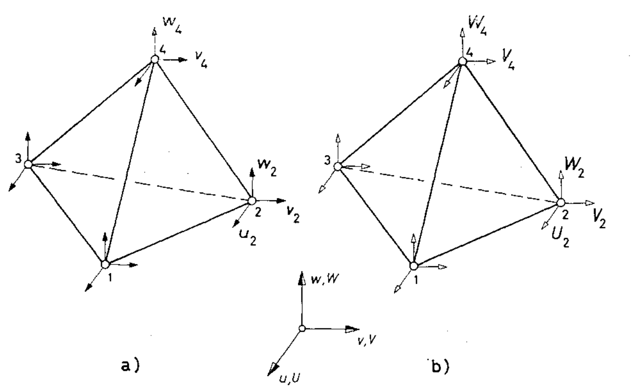

Fig. 4 Cartesian nodal displacements and forces.

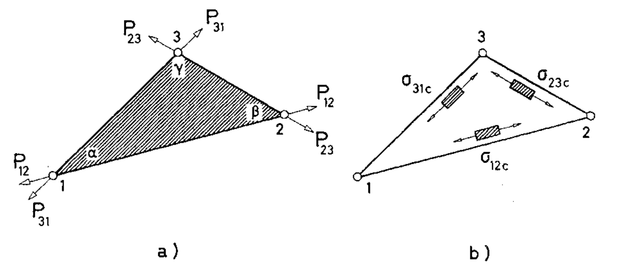

Fig. 5 Triangular element; natural stresses and loads.

1.2 Natural Stresses and Nodal Loads

In Ref. 2, we introduced the device of defining the stress vector for a triangular element not by Cartesian direct and shear stresses but by three stress *components*, taken parallel to the three sides. These "triangular" stresses form the (3 × 1) natural or invariant stress vector $\boldsymbol{\sigma}_N$. In accordance with the notation of Fig. 5, we write

$$\boldsymbol{\sigma}_N = \{\sigma_{12c}\ \sigma_{23c}\ \sigma_{31c}\} \tag{22}$$

We emphasized in our previous work that the $\sigma_{i,i+1,c}$ is *not* the total stress in the direction $(i, i+1)$, since each component stress contributes to the actual stress along the other two sides, unless one side is orthogonal to $(i, i+1)$. It is straightforward, however, to derive, via the stress transformation formulas, the *total* stresses σ_{12}, σ_{23}, σ_{31}, or for that matter, of any other system of direct and shear stresses, e.g., the principal stresses and directions, in terms of the stress vector $\boldsymbol{\sigma}_N$. For example, the total stress vector

$$\boldsymbol{\sigma} = \{\sigma_{12}\ \sigma_{23}\ \sigma_{31}\} \tag{23}$$

is given by

$$\boldsymbol{\sigma} = \mathfrak{A}\boldsymbol{\sigma}_N \tag{24}$$

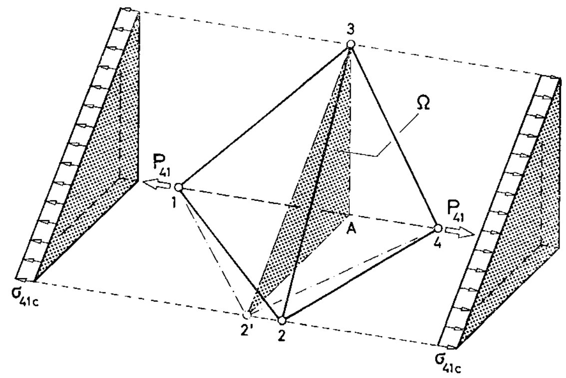

Fig. 6 Derivation of natural stress-nodal load relation.

where

$$\mathfrak{a} = \begin{bmatrix} 1 & \cos^2\beta & \cos^2\alpha \\ \cos^2\beta & 1 & \cos^2\gamma \\ \cos^2\alpha & \cos^2\gamma & 1 \end{bmatrix} \qquad (25)$$

and α, β, γ are the angles of the triangle (Fig. 5).

Similarly, the total direct stresses $\sigma_{i,i+1,n}$, normal to the three sides, and defined by the stress vector

$$\boldsymbol{\sigma}_n = \{\sigma_{12n}\ \sigma_{23n}\ \sigma_{31n}\} \qquad (26)$$

are obtained from

$$\boldsymbol{\sigma}_n = -\mathfrak{a}_n\boldsymbol{\sigma}_N \qquad (27)$$

where

$$\mathfrak{a}_n = \mathfrak{a} - \mathbf{e}_3\mathbf{e}_3^t \qquad (28)$$

Let us next consider the case of the tetrahedron. Here the natural stress vector must evidently be expressed in terms of the component stresses, taken parallel to the six edges and in the same order as in $\boldsymbol{\rho}_N$. Thus, $\boldsymbol{\sigma}_N$ becomes here

$$\boldsymbol{\sigma}_N = \{\sigma_{12c}\ \sigma_{23c}\ \sigma_{34c}\ \sigma_{41c}\ \sigma_{13c}\ \sigma_{24c}\} \qquad (29)$$

The (6×1) vector $\boldsymbol{\sigma}$ of the total stresses along the edges is again given by the matrix equation (24), but $\mathfrak{a}$ is here the (6×6) matrix,

$$\mathfrak{a} = \begin{bmatrix} 1 & c_{12,23}^2 & c_{12,34}^2 & c_{12,41}^2 & c_{12,13}^2 & c_{12,24}^2 \\ & 1 & c_{23,34}^2 & c_{23,41}^2 & c_{23,13}^2 & c_{23,24}^2 \\ & & 1 & c_{34,41}^2 & c_{34,13}^2 & c_{34,24}^2 \\ & & & 1 & c_{41,13}^2 & c_{41,24}^2 \\ & \text{symmetric} & & & 1 & c_{13,24}^2 \\ & & & & & 1 \end{bmatrix} \qquad (30)$$

In Eq. (30) we use for typographical reasons the compact notation

$$c_{ij,rs} = \cos(ij,\ rs) \qquad (30a)$$

To obtain the trigonometric values, we simply apply the operation

$$\cos(ii,\ rs) = \mathbf{c}_{ij}\mathbf{c}_{rs}^t \qquad (31)$$

We are now in a position to establish the matrix relationship between $\boldsymbol{\sigma}_N$ and the force vector $\mathbf{P}_N$. To this purpose, consider Fig. 6, in which it is assumed that the tetrahedron element is solely under the action of the component stress $\sigma_{41,c}$. It is clear that the volume V of the given tetrahedron is equal to that of the tetrahedron 12′34, where 2′ lies at the intersection of the plane through 3, normal to the edge (1, 4), and of a line through 2, parallel to (1, 4). It follows that the area Ω of the triangle (A, 2′, 3) is obtained from

$$\Omega = 3V/l_{41} \qquad (32)$$

V itself is given by

$$V = \tfrac{1}{6}|\mathbf{e}_4\ \boldsymbol{\phi}| \qquad (33)$$

where $\mathbf{e}_4$ is the unit column matrix of order 4 and

$$\boldsymbol{\phi} = \{\boldsymbol{\phi}_1\ \boldsymbol{\phi}_2\ \boldsymbol{\phi}_3\ \boldsymbol{\phi}_4\} \qquad (34)$$

is a (4×3) matrix [see Eq. (2)].

Simple statics show that the stress field σ_{41c} does not yield any statically equivalent nodal forces at 2 and 3. On the other hand, the equivalent pair of forces at 1 and 4 is evidently

$$P_{41} = \sigma_{41c}\Omega/3 = \sigma_{41c}V/l_{41} \qquad (35)$$

where the second relation derives from Eq. (32). Similar equations apply for the other five stress components. Hence, we deduce

$$\mathbf{P}_N = V\mathbf{l}_d^{-1}\boldsymbol{\sigma}_N \qquad (36)$$

or

$$\boldsymbol{\sigma}_N = (1/V)\mathbf{l}_d\mathbf{P}_N \qquad (36a)$$

where $\mathbf{l}_d$ is the diagonal matrix

$$\mathbf{l}_d = \lfloor l_{12}\ l_{23}\ l_{34}\ l_{41}\ l_{13}\ l_{24}\rfloor \qquad (37)$$

This completes the correlation of the vectors $\mathbf{P}_N$ and $\boldsymbol{\sigma}_N$.

1.3 Natural Strains and Stiffnesses

Assume that a component stress σ_{23c} is acting parallel to the edge (2, 3). It clearly gives rise to the strains

$$\sigma_{23c}/E \qquad -\nu\sigma_{23c}/E \qquad (a)$$

in the (2, 3) direction, and transverse radial direction, respectively. Consider next another edge direction, say (4, 1), at an angle

$$\alpha = (23,\ 41) \qquad (b)$$

to the edge (2, 3). The strains (a) produce the following total strain along (4, 1):

$$\epsilon_{41} = (\sigma_{23c}/E)(\cos^2\alpha - \nu\sin^2\alpha) = (\sigma_{23c}/E)[(1+\nu)\cos^2\alpha - \nu] \qquad (c)$$

Extending the argument to the other edge strains it is easy to see that the (6×1) natural strain matrix

$$\boldsymbol{\varepsilon}_N = \{\epsilon_{12}\ \epsilon_{23}\ \epsilon_{34}\ \epsilon_{41}\ \epsilon_{13}\ \epsilon_{24}\} \qquad (38)$$

derives from

$$\boldsymbol{\varepsilon}_N = (1/E)\mathfrak{a}_T\boldsymbol{\sigma}_N \qquad (39)$$

where $\mathfrak{a}_T$ is given by

$$\mathfrak{a}_T = (1+\nu)\mathfrak{a} - \nu\mathbf{e}_6\mathbf{e}_6^t \qquad (40)$$

Equation (39) may also be written as

$$\boldsymbol{\sigma}_N = \mathbf{E}_N\boldsymbol{\varepsilon}_N \qquad (41)$$

in which

$$\mathbf{E}_N = E\mathfrak{a}_T^{-1} \qquad (42)$$

The (6×6) (stiffness) matrix $\mathbf{E}_N$, which relates $\boldsymbol{\sigma}_N$ and $\boldsymbol{\varepsilon}_N$, may be denoted as the natural elasticity modulus.

The final step leading to the stiffness matrix of the tetrahedron is straightforward. The *total* elongation $\boldsymbol{\rho}_N$ of the edges may be expressed in terms of strains and stresses as follows:

$$\boldsymbol{\rho}_N = \mathbf{l}_d\boldsymbol{\varepsilon}_N = (1/E)\mathbf{l}_d\mathfrak{a}_T\boldsymbol{\sigma}_N \qquad (43)$$

Applying Eq. (36), we find

$$\boldsymbol{\rho}_N = (1/EV)\mathbf{l}_d\mathfrak{a}_T\mathbf{l}_d\mathbf{P}_N = \mathbf{f}_N\mathbf{P}_N \qquad (44)$$

where

$$\mathbf{f}_N = (1/EV)\mathbf{l}_d\mathfrak{a}_T\mathbf{l}_d \qquad (45)$$

is the so-called natural flexibility of the tetrahedron. Equation (44) may also be written as a stiffness relation

$$\mathbf{P}_N = \mathbf{k}_N \boldsymbol{\varrho}_N \tag{46}$$

where

$$\mathbf{k}_N = EV \mathbf{l}_d^{-1} \mathfrak{a}_T^{-1} \mathbf{l}_d^{-1} \tag{47}$$

is the required (6 × 6) symmetrical matrix for the natural stiffness of a tetrahedron. It should be noted that it is unnecessary and, in fact, computationally inexpedient to evaluate the explicit algebraic forms of the individual coefficients. In the presence of thermal effects, we may allow for the dependence of the modulus E on the temperature.

For the analysis of the assembled structure it is advantageous to refer the stiffness of each element to the complete set of Cartesian displacements at the nodal points. To obtain this expanded stiffness we premultiply Eq. (46) with $\mathbf{T}^t\mathbf{D}^t\mathbf{C}_N^t$ and substitute expression (20) for $\boldsymbol{\varrho}_N$. We find, using Eq. (21),

$$\mathbf{P} = \mathbf{k}\boldsymbol{\varrho} \tag{48}$$

where $\mathbf{k}$ is the Cartesian stiffness, of order 12 × 12,

$$\mathbf{k} = \mathbf{T}^t\bar{\mathbf{k}}\mathbf{T} = \mathbf{T}^t\mathbf{D}^t\mathbf{C}_N^t\mathbf{k}_N\mathbf{C}_N\mathbf{D}\mathbf{T} \tag{49}$$

In Eq. (49) $\bar{\mathbf{k}}$ is the stiffness based on the sequential order $\bar{\boldsymbol{\varrho}}$.

We now introduce the matrix of the stiffnesses of the unassembled constituent elements of the body. Assuming that there are s elements, we write

$$\mathbf{k} = \lceil \mathbf{k}_1\, \mathbf{k}_2\, \mathbf{k}_3 \ldots \mathbf{k}_p \ldots \mathbf{k}_s \rfloor \tag{50}$$

Equations (50) is a diagonal supermatrix of order $s \times s$, each element of which is a symmetrical (12 × 12) submatrix.

1.4 Stiffness of the Complete Structure; Displacements, Stresses

In what follows, all main formulas are based on a common Cartesian coordinate system $Oxyz$. We assume at first that there are no thermal actions on the body. Let $\mathbf{r}$ be the Cartesian displacement vector of order $(3n - t) \times 1$, where n is the number of nodal points and t that of supporting actions. The corresponding *external* force vector, also of dimensions $(3n - t) \times 1$, is denoted by $\mathbf{R}$. If $\mathbf{K}$ stands for the stiffness of the *assembled* structure, the load-displacement relation reads

$$\mathbf{R} = \mathbf{K}\mathbf{r} \tag{51}$$

In order to establish an elegant, and from the programing point of view, advantageous technique for the assembly of the stiffness $\mathbf{K}$, it is convenient to apply the congruent transformation expression of Ref. 5 to the matrix of Eq. (50) in the form

$$\mathbf{K} = \mathbf{a}^t\mathbf{k}\mathbf{a} \tag{52}$$

where the extremely sparsely populated matrix $\mathbf{a}$ is of dimensions $12s \times (3n - t)$. Formula (52) may be proved easily by noting that the $(s \times 1)$ displacement supervector $\boldsymbol{\varrho}$ formed by the $\boldsymbol{\varrho}_p$ vectors of the s elements,

$$\boldsymbol{\varrho} = \{\boldsymbol{\varrho}_1\, \boldsymbol{\varrho}_2 \ldots \boldsymbol{\varrho}_p \ldots \boldsymbol{\varrho}_s\} \tag{53}$$

[see Eq. (7)] must be related to the vector $\mathbf{r}$ by an expression of the type

$$\boldsymbol{\varrho} = \mathbf{a}\mathbf{r} \tag{54}$$

Application of the dual transformation rules for forces and displacements yields then Eq. (52). It is sometimes more advantageous to assemble $\mathbf{K}$ directly from the stiffness $\bar{\mathbf{k}}$ of Eq. (49).

In the present case, both $\mathbf{K}$ and $\mathbf{k}$ are referred to the same basic coordinate system $Oxyz$ and the matrix $\mathbf{a}$ is consequently seen to contain only "ones" as nonzero elements. Hence, the operations of Eq. (52) involve merely additions, and this simple process is easily performed by the algorithm proposed in Ref. 1, which only stores the *locations* of the "unit" elements of $\mathbf{a}$ and ignores all zeros. This underlines the inherent simplicity of the procedure. Reference 1 also discusses some additional devices to generate the "list" $\mathbf{a}$ by simple instructions in the computer.

The solution of Eq. (51),

$$\mathbf{r} = \mathbf{K}^{-1}\mathbf{R} \tag{51a}$$

yields the displacement $\mathbf{r}$. There follow for each element p the natural strains (suffix p omitted)

$$\boldsymbol{\varepsilon}_N = \mathbf{l}_d^{-1}\boldsymbol{\varrho}_N = \mathbf{l}_d^{-1}\mathbf{C}_N\mathbf{D}\bar{\boldsymbol{\varrho}} = \mathbf{l}_d^{-1}\mathbf{C}_N\mathbf{D}\mathbf{T}\boldsymbol{\varrho} = \mathbf{l}_d^{-1}\mathbf{C}_N\mathbf{D}\mathbf{T}\mathbf{a}\mathbf{r} \tag{55}$$

and hence, the natural stresses $\boldsymbol{\sigma}_N$ from Eq. (41). Finally, we may obtain the principal stresses or any other system of Cartesian stresses.

Alternatively, we may derive the Cartesian strain vector

$$\boldsymbol{\varepsilon} = \{\epsilon_{xx}\, \epsilon_{yy}\, \epsilon_{zz}\, \epsilon_{xy}\, \epsilon_{yz}\, \epsilon_{zx}\} \tag{56}$$

directly from the displacement vector $\boldsymbol{\varrho}$. The associated stress $\boldsymbol{\sigma}$ follows from the standard Cartesian stress-strain relations, but this method is not as neat as the first.

We now extend our analysis and assume the joint presence of loads and thermal actions giving rise to stresses.‡ Following Ref. 5, the equilibrium condition (51) takes the form

$$\mathbf{R} = \mathbf{K}\mathbf{r} + \mathbf{a}^t\mathbf{J} \tag{57}$$

where $\mathbf{a}$ is once more the identification matrix of Eq. (52), and $\mathbf{J}$ is the column matrix of the so-called *initial* nodal loads arising in each element through thermal action if all nodal displacements are blocked. The vector $\mathbf{J}$ is, of course, expressed in the Cartesian components of the nodal loads and follows the same sequential order as $\boldsymbol{\varrho}$ of Eq. (53). For the s elements we may write $\mathbf{J}$ as the $(s \times 1)$ super-vector

$$\mathbf{J} = \{\mathbf{J}_1\, \mathbf{J}_2 \ldots \mathbf{J}_p \ldots \mathbf{J}_s\} \tag{58}$$

where each submatrix $\mathbf{J}_p$ is of order 12 × 1.

To find the typical subvector $\mathbf{J}_p$, it is again convenient to proceed via an intermediate "natural" definition of the initial loads in an element. Hence, the entries of this natural thermal vector $\mathbf{J}_{Np}$ are the initial loads at the nodal points measured in the directions of the edges. To determine $\mathbf{J}_{Np}$, we first obtain the (6 × 1) natural vector $\boldsymbol{\varrho}_{N\Theta p}$, which reproduces the *free* thermal expansions of the six edges, from the obvious relation (suffix p omitted)

$$\boldsymbol{\varrho}_{N\Theta} = \eta\mathbf{l} \tag{59}$$

where $\eta = \alpha\Theta$ is the free thermal strain,§ assumed uniform in the element, and

$$\mathbf{l} = \{l_{12}\, l_{23}\, l_{34}\, l_{41}\, l_{13}\, l_{24}\} \tag{60}$$

is the (6 × 1) column matrix of the lengths of the edges [see also Eq. (37)]. The vector $\mathbf{J}_N$ is now deduced from the condition that the corresponding elastic elongation vector suppresses the thermal expansion vector $\boldsymbol{\varrho}_{N\Theta}$. Thus, using the load-displacement Eq. (46), we have

$$\mathbf{J}_N = -\mathbf{k}_N\boldsymbol{\varrho}_{N\Theta} = -\eta\mathbf{k}_N\mathbf{l} \tag{61}$$

To establish the vector $\mathbf{J}$, we must evidently apply the same transformation as in the second of Eqs. (21). Therefore, we find for each element

$$\mathbf{J} = \mathbf{T}^t\mathbf{D}^t\mathbf{C}_N^t\mathbf{J}_N = -\eta\mathbf{T}^t\mathbf{D}^t\mathbf{C}_N^t\mathbf{k}_N\mathbf{l} \tag{62}$$

It is clear that the column vector $\mathbf{J}$ may alternatively be expressed in terms of the Cartesian coordinates of the nodal

‡ The theory is, of course, equally applicable to any other type of initial stresses, e.g., those due to lack of fit.

§If α varies with Θ, η is given by $\int \alpha d\Theta$.

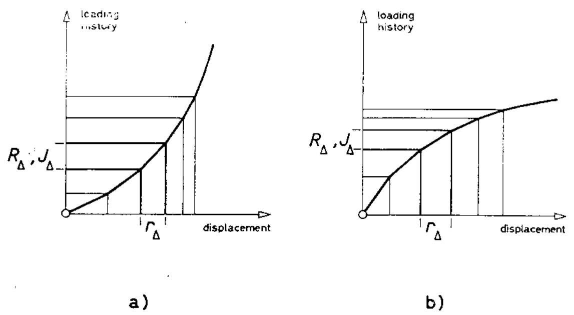

Fig. 7 Piecewise linearization (incremental or iteration technique) for large displacement (nonlinear) analysis.

points. For this purpose we introduce the two (12 × 1) column matrices

$$\left.\begin{aligned} \boldsymbol{\psi} &= \{\mathbf{x}\ \mathbf{y}\ \mathbf{z}\} \\ \bar{\boldsymbol{\psi}} &= \mathbf{T}\boldsymbol{\psi} = \{\boldsymbol{\psi}_1\ \boldsymbol{\psi}_2\ \boldsymbol{\psi}_3\ \boldsymbol{\psi}_4\} \end{aligned}\right\} \tag{63}$$

where $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are (4 × 1) submatrices of the type

$$\mathbf{x} = \{x_1\ x_2\ x_3\ x_4\} \text{ etc.}$$

and $\boldsymbol{\psi}_1$ to $\boldsymbol{\psi}_4$ are defined in Eq. (2). Using now the transformation rule of Eq. (20), we derived the physically evident formulas

$$\bar{\mathbf{J}} = -\eta\bar{\mathbf{k}}\bar{\boldsymbol{\psi}} \qquad \text{and} \qquad \mathbf{J} = \mathbf{T}^t\bar{\mathbf{J}} = -\eta\mathbf{k}\boldsymbol{\psi} \tag{64}$$

Although Eqs. (64) are mathematically very simple, relation (62) proves, in general, computationally more advantageous.

Since the selected idealization satisfies the compatibility conditions everywhere, but the equilibrium conditions only at the nodal points, our procedure, whilst exact in the limit, underestimates the flexibility of the system corresponding to a load R.

2. Large Displacement Analysis

2.1 Fundamentals

As demonstrated in Ref. 1, the specification of the natural force and displacement vectors allows a concise extension of the basic theory to account for the geometrical effects of large displacements. Since we must presume that the reader is familiar with the main train of thought, we only present here an apercu of the basic theory. We recall that the procedure starts by approximating the loading by an appropriate sequence of simultaneous or separate force and thermal loading steps $\mathbf{R}_\Delta$ and $\mathbf{J}_\Delta$ (see Figs. 7a and 7b). The corresponding incremental displacement vectors $\mathbf{r}_\Delta$ are then calculated from the standard linear equation (57), rewritten in the current notation as

$$\mathbf{R}_\Delta = \mathbf{K}\mathbf{r}_\Delta + \mathbf{a}^t\mathbf{J}_\Delta \tag{65}$$

where the "stiffness" $\mathbf{K}$ of the body consists, however, of two component matrices

$$\mathbf{K} = \mathbf{K}_E + \mathbf{K}_G = \mathbf{a}^t\mathbf{k}_E\mathbf{a} + \mathbf{a}^t\mathbf{k}_G\mathbf{a} \tag{66}$$

Here $\mathbf{k}_E$ (previously denoted by $\mathbf{k}$) is the elastic stiffness matrix of the unassembled elements, strictly based on the instantaneous geometry and temperature of each element, and $\mathbf{k}_G$ is the additional, so-called geometrical stiffness of the elements. Now, Ref. 1 shows that the stiffness $\mathbf{k}_G$ is linearly dependent on the components of the natural force vector $\mathbf{P}_N$ built up on the preceding loading steps. The reader should remember that this dependence of $\mathbf{k}_G$ is exclusively in terms of the natural force vector, this being one further proof of the intrinsic importance of this concept. At the same time, the geometry, not the contents of an element, determines its geometrical stiffness. A few further aspects of the theory are worth noting. Since the equilibrium relation between nodal forces and stresses [Eq. (36)] is based at each step on the instantaneous geometry, the stresses are the so-called "true" stresses referred to the current dimensions. Furthermore, the specification of the elastic stiffness includes the change of geometry and is allied to the definition of strain as the ratio of an incremental elongation to the corresponding parent length at the initiation of each step. Thus, the assumed linearity of (true) stress and elastic strain is only called for with respect to each increment. Otherwise, the stress-strain law may take any nonlinear form. In this connection, we also note that the theory is valid for large total strains as well as large deformations. Naturally, at the onset of truly large strains, an initially isotropic and homogeneous body becomes inhomogeneous and possibly also anisotropic. Inhomogeneity of the material from element to element is easily accounted for. Anisotropy (and nonelastic behavior) are covered in a subsequent publication.[9] Interesting applications of our theory occur in rubber technology and other fields and confirmatory experiments are already concluded.

We have to ignore in this apercu the technicalities of setting up $\mathbf{k}_{NE}$ (anew, if necessary, at each step). More interesting is the determination of $\mathbf{k}_G$ for an arbitrary tetrahedron. Before proceeding to its derivation, we establish in the following paragraph the vector $\mathbf{P}_N$ accumulated in a typical element in the antecedent loading steps.

2.2 The Vector $\mathbf{P}_N$

To obtain $\mathbf{P}_N$ we simply note that

$$\mathbf{P}_N = \Sigma\, \mathbf{P}_{N\Delta} \tag{67}$$

where the summation extends over the preceding loading increments. Since temperature effects are taken to be present, the incremental vector $\mathbf{P}_{N\Delta}$ derives from the elastic component $\boldsymbol{\varrho}_{N\Delta E}$ of the natural vector $\boldsymbol{\varrho}_{N\Delta}$. In fact,

$$\mathbf{P}_{N\Delta} = \mathbf{k}_{NE}\boldsymbol{\varrho}_{N\Delta E} \tag{68}$$

Clearly

$$\boldsymbol{\varrho}_{N\Delta E} = \boldsymbol{\varrho}_{N\Delta} - \eta_\Delta \mathbf{l} \tag{69}$$

where $\mathbf{l}$ is the column matrix of Eq. (60). The vector $\boldsymbol{\varrho}_{N\Delta}$ itself is obtained from $\mathbf{r}_\Delta$ by a transformation contained in Eq. (55). Using Eqs. (68) and (69) in Eq. (67) we find

$$\mathbf{P}_N = \Sigma\, \mathbf{k}_{NE}\boldsymbol{\varrho}_{N\Delta} - \Sigma\, \eta_\Delta\mathbf{k}_{NE}\mathbf{l} = \Sigma\, \mathbf{k}_{NE}\mathbf{C}_N\mathbf{D}\mathbf{T}\boldsymbol{\varrho}_\Delta - \Sigma\, \eta_\Delta\mathbf{k}_{NE}\mathbf{l} \tag{70}$$

which completely determines the vector $\mathbf{P}_N$ for each element.

2.3 Geometrical Stiffness $\mathbf{k}_G$

We are now in a position to establish the geometrical stiffness of the tetrahedron. Following our statement in 2.1 and the detailed argument for a triangular element in Refs. 1 and 2, we observe (Fig. 8) that the present problem is identical to that of a pin-jointed six-bar framework whose members form a tetrahedron, geometrically identical to the given continuous one, and are subjected to constant loads P_{12} to P_{24} (Fig. 7). Hence, it effectively suffices to establsih the geometrical stiffness of a single bar. To this purpose, consider Fig. 8a, which shows a bar (i, j) in equilibrium under a force P_{ij}. The (6 × 1) Cartesian force and displacement vectors $\bar{\mathbf{P}}$, $\bar{\boldsymbol{\varrho}}$ are written as

$$\bar{\mathbf{P}} = \{\mathbf{P}_i\ \mathbf{P}_j\} \qquad \bar{\boldsymbol{\varrho}} = \{\boldsymbol{\varrho}_i\ \boldsymbol{\varrho}_j\} \tag{71}$$

where $\mathbf{P}_i$, $\boldsymbol{\varrho}_i$ etc. are defined in Eqs. (7) and (8). Using the row matrix $\mathbf{c}$ of Eq. (16) for the direction cosines, we obtain by inspection

$$\bar{\mathbf{P}} = P_{ij}\mathbf{c}_e^{\,t} \tag{72}$$

$$\mathbf{c}_e = [-\mathbf{c}_{ij}\ \mathbf{c}_{ij}] \tag{73}$$

Any change in the direction cosines (but not in the length of the bar!) due to nodal displacements $\bar{\boldsymbol{\rho}}_\Delta$ leads to an incremental $\bar{\mathbf{P}}_\Delta$ which is given by

$$\bar{\mathbf{P}}_\Delta = P_{ij}\mathbf{c}_{e\Delta}{}^t \tag{74}$$

An elementary argument yields

$$\mathbf{c}_{e\Delta}{}^t = \mathbf{g}_{ij}\bar{\boldsymbol{\rho}}_\Delta \tag{75}$$

where

$$\mathbf{g}_{ij} = \frac{1}{l_{ij}}\left\{\begin{bmatrix} \mathbf{I}_3 & -\mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix} - \mathbf{c}_e{}^t\mathbf{c}_e\right\} \tag{76}$$

Substitution of Eq. (75) into (74) leads to

$$\bar{\mathbf{P}}_\Delta = \bar{\mathbf{k}}_{Gij}\bar{\boldsymbol{\rho}}_\Delta \tag{77}$$

Here, $\bar{\mathbf{k}}_{Gij}$, the required geometrical stiffness of a bar, is simply (see also Ref. 1)

$$\bar{\mathbf{k}}_{Gij} = P_{ij}\mathbf{g}_{ij} \tag{78}$$

Having established the six stiffnesses $\bar{\mathbf{k}}_{G12}$, $\bar{\mathbf{k}}_{G23}$, $\bar{\mathbf{k}}_{G34}$, $\bar{\mathbf{k}}_{G41}$, $\bar{\mathbf{k}}_{G13}$, and $\bar{\mathbf{k}}_{G24}$, of the bars, the (12 × 12) geometrical stiffness $\bar{\mathbf{k}}_G$ of the tetrahedron based on the displacement vector $\bar{\boldsymbol{\rho}}_\Delta$ is easily assembled by congruent transformations similar to those introduced in Ref. 1. Thus,

$$\bar{\mathbf{k}}_G = \sum_{ij=1,2}^{2,4} \mathbf{M}_{ij}{}^t\bar{\mathbf{k}}_{Gij}\mathbf{M}_{ij} \tag{79}$$

where $\mathbf{M}_{ij}$ are (2 × 4) location supermatrices, each element of which is either a unit or a zero matrix of order 3. Their rule of formation is that the unit matrix $\mathbf{I}_3$ appears only in the ith "element" of the first row and the jth "element" of the second row. For example,

$$\mathbf{M}_{12} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 \\ \mathbf{O}_3 & \mathbf{I}_3 & \mathbf{O}_3 & \mathbf{O}_3 \end{bmatrix}$$
$$\mathbf{M}_{24} = \begin{bmatrix} \mathbf{O}_3 & \mathbf{I}_3 & \mathbf{O}_3 & \mathbf{O}_3 \\ \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{O}_3 & \mathbf{I}_3 \end{bmatrix} \tag{80}$$

Since we need, in general, the sequential order $\mathbf{k}_G$ for the displacement vector $\boldsymbol{\rho}_\Delta$, whose sequence is that of Eq. (7), we write

$$\mathbf{k}_G = \mathbf{T}^t\bar{\mathbf{k}}_G\mathbf{T} \tag{81}$$

where $\mathbf{T}$ is given by Eq. (14). This concludes the analysis of the large displacement problem. For the special case of primary or secondary instability, the reader is referred to Ref. 1 or 2.

Appendix: Dynamic Phenomena

The extension of our theory to dynamic phenomena is straightforward. Consider, for example, the case of free oscillations. We first write the *components* of the idealized nodal masses of the s tetrahedra as the (12s × 1) column matrix

$$\mathbf{m} = (\rho/4)\{V_1\mathbf{e}_{12}\ V_2\mathbf{e}_{12}\ldots V_p\mathbf{e}_{12}\ldots V_s\mathbf{e}_{12}\} \tag{82}$$

where ρ is the density of the material. In Eq. (82) the eftive masses are separately listed for the x, y, z directions. We next assemble the *total* masses, acting in each of the free (i.e., unsupported) nodal directions, in the form of the $(3n - t) \times 1$ column matrix

$$\mathbf{M} = \mathbf{a}^t\mathbf{m} \tag{83}$$

Note that the applied transformation is dual to that of Eq. (54). The displacement vector $\mathbf{r}$ may now be represented as

$$\mathbf{r} = \mathbf{r}_0 e^{i\omega t} \tag{84}$$

where $\mathbf{r}_0$ is the $(3n - t) \times 1$ (eigen) vector of the amplitude of the displacements and ω is the circular frequency. The

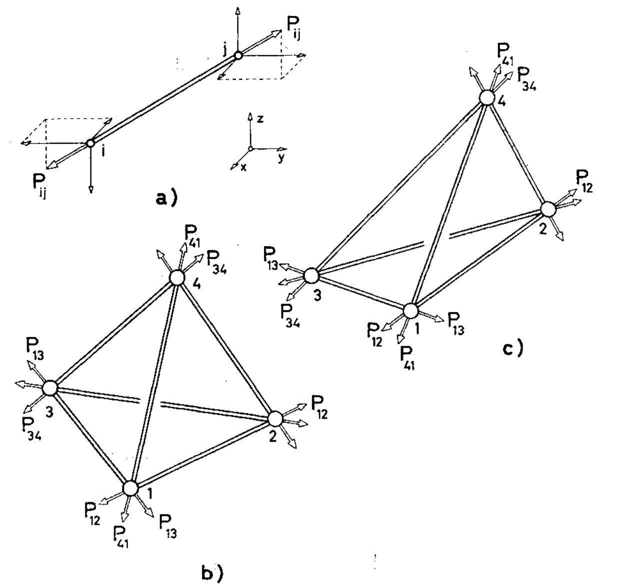

Fig. 8 a) Bar under end load P_{ij}, b) tetrahedron element at initial state, and c) tetrahedron at final (displaced and deformed) state of current increment. Applied natural loads are seen to be invariant, if elastic component loads are ignored.

inertia loads $\mathbf{R}_i$ hence may be established by the matrix operation

$$\mathbf{R}_i = -\mathbf{M}_d\ddot{\mathbf{r}} = e^{i\omega t}\omega^2\mathbf{M}_d\mathbf{r}_0 \tag{85}$$

Here $\mathbf{M}_d$ reads $\mathbf{M}$ as a diagonal matrix of order $(3n - t) \times (3n - t)$. Substituion of Eqs. (89) and (85) in Eq. (51) yields

$$[\mathbf{M}_d{}^{-1}\mathbf{K} - \omega^2\mathbf{I}]\mathbf{r}_0 = \mathbf{O} \tag{86a}$$

or

$$[\mathbf{K}^{-1}\mathbf{M}_d - (1/\omega^2)\mathbf{I}]\mathbf{r}_0 = \mathbf{O} \tag{86b}$$

Equations (86a) and (86b) may be used, in conjunction with standard programs, to yield the eigenvectors and eigenvalues.

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